

RATIONAL SINGULARITIES AND UNIFORM SYMBOLIC TOPOLOGIES

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ABSTRACT. Take (R, \mathfrak{m}) any normal Noetherian domain, either local or \mathbb{N} -graded over a field. We study the question of when R satisfies the uniform symbolic topology property (USTP) of Huneke, Katz, and Validashti: namely, that there exists an integer $D > 0$ such that for all prime ideals $P \subseteq R$, the symbolic power $P^{(Da)} \subseteq P^a$ for all $a > 0$. Reinterpreting results of Lipman, we deduce that when R is a two-dimensional rational singularity, then it satisfies the USTP. Emphasizing the non-regular setting, we produce explicit, effective multipliers D , working in two classes of surface singularities in equal characteristic over an algebraically closed field, using: (1) the volume of a parallelogram in \mathbb{R}^2 when R is the coordinate ring of a simplicial toric surface; or (2) known invariants of du Val isolated singularities in characteristic zero due to Lipman.

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1. INTRODUCTION AND CONVENTIONS FOR THE PAPER

Given a Noetherian ring R , an important open problem is discerning when there is an integer $D = D(R) > 0$, depending on R , such that the symbolic power $P^{(Da)} \subseteq P^a$ for all prime ideals $P \subseteq R$ and all integers $a > 0$. When such a D exists, the P -adic and P -symbolic topologies on R are said to be **uniformly** linearly equivalent **for all primes** P , and borrowing from ([17]), we say that R satisfies the **uniform symbolic topology property (USTP)** on prime ideals. If inclined, one may pose the same problem for symbolic powers of radical (or even arbitrary)

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ideals; see Section (2) for definitions. Since inclusion between two ideals is a local property, one may reduce checking that a particular candidate D works to the case where (R, \mathfrak{m}) is local.

There are several simple cases where $D = 1$ will suffice, namely if R is: (a) Artinian; or (b) a one-dimensional domain; or (c) a two-dimensional UFD ($\text{Cl}(R) = 0$ in Lemma (1.1) below). In particular, being UFD's $D = 1$ works for any regular local ring of dimension at most two, and hence for any regular ring of dimension at most two. In view to case (a), we typically assume that R has positive Krull dimension.

Going further, in its simplest form, the Ein-Lazarsfeld-Smith Theorem ([6, 13]), as extended by Hochster and Huneke, says that if R is a d -dimensional Noetherian regular ring containing a field, then $P^{(da)} \subseteq P^a$ for all prime ideals $P \subseteq R$ and all integers $a > 0$; the papers ([14, 18, 23]) extend what is known in the setting of Noetherian regular rings containing a field. In stark contrast, effective uniform bounds are harder to unearth in the non-regular setting. In this direction, Huneke, Katz, and Validashti ([16]) show that a uniform D exists when R is a reduced isolated singularity which either contains a field of positive characteristic and is F -finite, or is essentially of finite type over a field of characteristic zero; however, their arguments are non-constructive, and as far as the author knows, no constructive (or even sharp) bounds D were known prior to cases covered in this paper.

The following is a first approximation of the key lemma we use to produce effective bounds:

Lemma 1.1. *Let R be a Noetherian normal domain of positive Krull dimension whose global divisor class group $\text{Cl}(R) := \text{Cl}(\text{Spec}(R))$ is finite abelian of order D . Then for all ideals $\mathfrak{q} \subseteq R$ of pure height one, the symbolic power $\mathfrak{q}^{(Da)} \subseteq \mathfrak{q}^a$ for all $a > 0$.*

Analogous with the uniformity results of ([6, 13, 16]), Lemma (1.1) can be adduced in support of Huneke's philosophy in ([15]) that there are uniform bounds lurking in commutative algebra. In particular, when R as in Lemma (1.1) has Krull dimension two, it satisfies the USTP on prime ideals; indeed, if P is the zero ideal or maximal, then $P^{(n)} = P^n$ for all $n > 0$, while symbolic powers of height one primes can have nontrivial behavior under the inclusions we study. Notice however that, in contrast with the Ein-Lazarsfeld-Smith and Huneke-Katz-Validashti results, the R in Lemma (1.1) is allowed to be of **mixed characteristic**, meaning that it is a ring of characteristic zero such that for some ideal $I \subsetneq R$, the quotient R/I has positive characteristic.

It remains to specify classes of rings (aside from Noetherian UFDs) that are known to satisfy the hypotheses of Lemma (1.1). In dimension two, we situate the following result due to Lipman:

Theorem (Lipman ([19], Proposition 17.1)). *Let (R, \mathfrak{m}) be a two-dimensional, normal Noetherian local domain. If R has a rational singularity, then $\text{Cl}(R)$ is finite.*

A two-dimensional, normal Noetherian local domain (R, \mathfrak{m}) has a **rational singularity** if there is a proper, birational map $f: X \rightarrow \text{Spec}(R)$ out of a regular scheme X such that $H^1(X, \mathcal{O}_X) = 0$; this definition comes from Lipman ([19], §1). Our lemma then affords us with a

Corollary 1.2. *All two-dimensional rational singularities (R, \mathfrak{m}) satisfy the USTP on primes.*

In ([17], Question 1.1), Huneke, Katz, and Validashti ask if every complete local domain R satisfies the USTP on primes; in the spirit of Swanson ([22], Theorem 3.3), they show in ([17], Proposition 2.4) that for each prime ideal $P \subseteq R$ in a complete local domain, there is a $D > 0$ depending on P such that $P^{(Da)} \subseteq P^a$ for all integers $a > 0$. Per the corollary, Question 1.1 has a positive answer when R is a two-dimensional rational singularity. However, we want concrete values D

in some other non-regular cases. Fortunately, when R is a complete du Val (ADE) isolated singularity in equal characteristic zero, with algebraically closed residue field, Lipman computes the order of $\text{Cl}(R)$ explicitly (cf., [19], §24). Thus we can identify concrete, uniform symbolic topology multipliers $D > 0$ in du Val singularities, complementing the main results of ([16, 17]). However, as we will consider in subsection (3.2), the smallest possible bound $D = D_{\min}(R)$ for satisfying the USTP **need not** be susceptible to description via a formula in terms of the ring's simplest numerical invariants, such as Krull dimension or the multiplicity at an isolated singularity, contrary to what experts might expect.

To produce effective bounds in higher dimension, we turn to the class of simplicial toric rings containing an arbitrary algebraically closed field k . A (simplicial) toric ring over k is the coordinate domain R of a (simplicial) normal affine toric variety X over k ; in particular, R is a normal semigroup k -algebra. For all intents and purposes, a simplicial toric ring is built from a convex-geometric starter: precisely, a polyhedral cone $\sigma \subseteq \mathbb{R}^n$ generated by a set of \mathbb{R} -linearly independent lattice points $v_i \in \mathbb{Z}^n$, each having coordinates whose gcd is one. It turns out that a toric ring has finite class group precisely when it is simplicial (cf., Theorem (3.1)(2) below), and in the following case, the class group's order admits a nice convex-geometric interpretation.

Theorem 1.3. *If R is a simplicial toric ring, and the v_i form an \mathbb{R} -basis, then the multiplier D in Lemma (1.1) is simply the volume of the n -parallelotope in \mathbb{R}^n spanned by v_1, \dots, v_n .*

In particular, this volume can be computed as the determinant of a positive-definite integer matrix. We prove Theorem (1.3) in subsection (3.1), using a short exact sequence.

We now outline the remainder of the paper. In Section (2), we prove the key lemma of the paper (Lemma (2.3)). In Section (3), we follow up on obtaining concrete uniform bounds in the toric- and du Val singularity settings. Finally, Section (4) situates lingering questions.

Conventions: All our rings are commutative with identity. From subsection (2.1) onwards, and except when stated otherwise, all our rings R will be (non-Artinian) Noetherian normal domains, and k will denote an algebraically closed field. All our algebraic varieties over k are irreducible. When we say R is **graded local**, we mean that $R = \bigoplus_{d \geq 0} R_d$ can be graded by \mathbb{N} , with R_0 being a field, and $\mathfrak{m} = \bigoplus_{d > 0} R_d$ the unique homogeneous maximal ideal.

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2. LOCAL UNIFORM ANNIHILATION AND SYMBOLIC POWERS

Defining Symbolic Powers. If P is a prime ideal in a nonzero Noetherian ring R , its **a -th symbolic power** ($a \in \mathbb{Z}_{>0}$) (Atiyah-Macdonald [2], Chapter 4, Exercise 13)

$$P^{(a)} := \{r \in R : ur \in P^a \text{ for some } u \in R - P\}$$

is the unique P -primary component in any minimal primary decomposition of P^a . More generally, if I is any proper ideal of R , and $\text{Ass}_R(R/I)$ is the set of associated primes of I , we define its **a -th symbolic power** $I^{(a)}$ by the rule:

$$f \in I^{(a)} \iff sf \in I^a \text{ for some } s \in S := R - \left(\bigcup_{P \in \text{Ass}_R(R/I)} P \right) = \bigcap_{P \in \text{Ass}_R(R/I)} (R - P).$$

According to ([2], Proposition 4.9), note that $I^{(1)} = I$ for any proper ideal $I \subseteq R$. In general, $I^{(a)} \supseteq I^a$ for all $a > 1$.

Take R the coordinate ring of a nonsingular affine variety X over an arbitrary algebraically closed field k . Then we may alternatively understand symbolic powers of a radical ideal I in terms of functions that vanish on the zero locus $Z = \mathbb{V}(I) \subseteq X$. For $x \in X$, let $\mathfrak{m}_x \subseteq R$ be the maximal ideal of regular functions that vanish at x . If $f \in R$ is a nonzero regular function vanishing at x , there's a unique $e \in \mathbb{Z}_{>0}$ such that $f \in \mathfrak{m}_x^e \setminus \mathfrak{m}_x^{e+1}$; we say f vanishes at x to order e , and let $\text{ord}_x(f) := e$ denote the **order of vanishing at f at x** . By convention, $\text{ord}_x(0) = \infty$. Zariski and Nagata showed (cf., Eisenbud and Hochster [7], proof of Corollary 1) that

$$\begin{aligned} I^{(a)} &= \{f \in R : \text{ord}_x(f) \geq a \text{ for all } x \in Z\} \\ &= \{f \in R : f \in \mathfrak{m}_x^a \text{ for all } x \in Z\} \\ &= \bigcap \{\mathfrak{m}^a : \mathfrak{m} \supseteq I \text{ is any maximal overprime of } I\}. \end{aligned}$$

Evidently, $I^{(a)} \supseteq I^a$ for all a , but in general the inclusion can be strict.

2.1. Uniform Annihilation and Class Groups. Given a normal Noetherian domain R , the **divisor class group** $\text{Cl}(R) = \text{Cl}(\text{Spec}(R))$ of R is the free abelian group on the set of height one prime ideals of R modulo relations of the form

$$(!) \quad a_1 P_1 + \dots + a_r P_r = 0,$$

whenever the ideal $P_1^{(a_1)} \cap \dots \cap P_r^{(a_r)}$ is principal (see Hochster's lecture notes [12] for more details on this definition; cf., Hartshorne's presentation [9], Ch.II, §6 for how to account for relations (!) where the a_i are allowed to be negative). Additionally, we define the *trivial* effective Weil divisor $\text{div}(\langle 1 \rangle R) = \text{div}(R) = [R] := 0$ of the unit ideal to have identically zero \mathbb{Z} -coefficients. According to Proposition 6.2 in ([9], Ch.II, §6), the following two conditions are equivalent:

- (1) $\text{Cl}(R) = 0$;
- (2) R is a UFD (that is, every height-one prime ideal is principal);

Moreover, (2) is equivalent to (3): All symbolic powers of a height-one prime ideal $P \subseteq R$ are principal. In particular, $P^{(a)} = P^a$ for all height-one primes $P \subseteq R$ in a UFD. Although Hartshorne does not include condition (3), that (2) and (3) are equivalent is immediate from the following fact: If $I = (f)R$ is a nonzero \mathfrak{p} -primary ideal in a domain R , then so is I^a for all integers $a > 0$. Induce on a , with base case $a = 1$. Suppose $\text{Ass}_R(R/I^a) = \{\mathfrak{p}\}$. Given the short exact sequence of R -modules,

$$0 \rightarrow I^a/I^{a+1} \rightarrow R/I^{a+1} \rightarrow R/I^a \rightarrow 0,$$

we have that $\emptyset \neq \text{Ass}_R(R/I^{a+1}) \subseteq \text{Ass}_R(I^a/I^{a+1}) \cup \text{Ass}_R(R/I^a)$ (cf., Thm 6.3 of Matsumura [20]). We show that $I^a/I^{a+1} \cong R/I$ as R -modules, so $\text{Ass}_R(I^a/I^{a+1}) = \text{Ass}_R(R/I) = \{\mathfrak{p}\}$. The R -linear surjection $\varphi: R \twoheadrightarrow I^a/I^{a+1} [x \mapsto x f^a]$ has kernel I . Indeed, if $x \in \ker \varphi$, then $x f^a \in I^{a+1}$, that is, $x f^a = y f^{a+1} = (y f) f^a$ for some $y \in R$. Thus $(x - y f) f^a = 0$ implies $x - y f = 0$, for $f^a \neq 0$ is a nonzerodivisor, since $f \neq 0$ in the domain R . Therefore, $x = y f \in I$. Thus $\emptyset \neq \text{Ass}_R(R/I^{a+1}) \subseteq \text{Ass}_R(R/I) \cup \text{Ass}_R(R/I^a) = \{\mathfrak{p}\} \cup \{\mathfrak{p}\} = \{\mathfrak{p}\}$ completing the induction.

Given a reduced Noetherian ring R , recall that the affine scheme $X = \text{Spec}(R)$ is **locally factorial** if $R_{\mathfrak{p}}$ is a UFD for all prime ideals $\mathfrak{p} \subseteq R$; or equivalently, if $\text{Cl}(R_{\mathfrak{p}}) = 0$ for all primes $\mathfrak{p} \subseteq R$. To generalize the class group formulation, we make a

Definition 2.1. Take a reduced Noetherian ring R such that $R_{\mathfrak{p}}$ is a normal domain for all prime ideals $\mathfrak{p} \subseteq R$. We say R is **(locally) uniformly annihilated** if there exists an integer (multiplier) $D > 0$ such that one, and hence all, of the following equivalent conditions will hold:

- (1) $D \cdot \text{Cl}(R_{\mathfrak{p}}) = 0$ for all prime ideals $\mathfrak{p} \subseteq R$. More precisely, $P^{(D)}R_{\mathfrak{p}} = (PR_{\mathfrak{p}})^{(D)}$ is principal for all height one primes $P \subseteq \mathfrak{p}$.
- (2) The annihilator ideal $\text{Ann}_{\mathbb{Z}}(\text{Cl}(R_{\mathfrak{p}})) \subseteq D\mathbb{Z}$ for all prime ideals $\mathfrak{p} \subseteq R$.
- (3) $D \cdot \text{Cl}(R_{\mathfrak{m}}) = 0$ for all maximal ideals $\mathfrak{m} \subseteq R$.

Notice that (3) implies (1) since the extension IS of a principal ideal $I \subseteq R$ along a ring homomorphism $\phi: R \rightarrow S$ is principal; in our case, we work with the localization maps $R_{\mathfrak{m}} \rightarrow (R_{\mathfrak{m}})_{\mathfrak{p}R_{\mathfrak{m}}} \cong R_{\mathfrak{p}}$ for all maximal ideals $\mathfrak{m} \subseteq R$ and all primes $\mathfrak{p} \subseteq \mathfrak{m}$ (cf., Corollary 11.28 of Altman-Kleiman [1] for the isomorphism). Notice also that $D = 1$ works if and only if R is locally factorial, and that it suffices to compute D relative to those maximal ideals $\mathfrak{m} \subseteq R$ such that $R_{\mathfrak{m}}$ is not a UFD.

When R is a Noetherian normal domain, and the annihilator ideal $\text{Ann}_{\mathbb{Z}}(\text{Cl}(R)) = D\mathbb{Z} \neq 0$ (e.g., if $\text{Cl}(R)$ is finite), R is uniformly annihilated by D . However, the smallest D that works need not annihilate $\text{Cl}(R)$ globally. For example, if the Dedekind domain $R = \mathbb{Z}[\sqrt{-5}]$, then $\text{Cl}(R) \cong \mathbb{Z}/2\mathbb{Z}$, while any Dedekind domain R is locally factorial ($D = 1$ works), since a discrete valuation ring is a PID, hence a UFD.

Remark. When R is normal Noetherian domain, algebraic geometers may prefer to process Definition (2.1) as a uniformly bounded \mathbb{Q} -Cartier property on $X := \text{Spec}(R)$. A Weil divisor $E = a_1P_1 + \cdots + a_rP_r$ on X is **Cartier** (locally principal) if

$$a_1[P_1R_{\mathfrak{m}}] + \cdots + a_r[P_rR_{\mathfrak{m}}] = 0 \in \text{Cl}(R_{\mathfrak{m}})$$

for all maximal ideals $\mathfrak{m} \subseteq R$. E is **\mathbb{Q} -Cartier** if the divisor $n \cdot E := (na_1)P_1 + \cdots + (na_r)P_r$ is Cartier for some integer $n > 0$; if so, the smallest n that works is the **index** of E . Since $X = \text{Spec}(R)$ is a normal Noetherian scheme, we can identify its **Picard group** $\text{Pic}(X)$ with the abelian group of Cartier divisors on X modulo linear equivalence (see Hartshorne [9], II.6 for omitted definitions and details). In particular, $\text{Pic}(X)$ is a subgroup of $\text{Cl}(X)$. Fix a *nontrivial* effective divisor $E = a_1P_1 + \cdots + a_rP_r$ on X , meaning all $a_i \geq 0$ and *some* $a_j > 0$. We note that by translating definitions of ([9], II.6) into commutative algebra, (!): E is Cartier if and only if the ideal $P_1^{(a_1)}R_{\mathfrak{m}} \cap \cdots \cap P_r^{(a_r)}R_{\mathfrak{m}}$ is principal for all maximal ideals $\mathfrak{m} \subseteq R$; if so, then equivalently the divisor class of E belongs to $\text{Pic}(X)$.

We define the **local class group** of X to be the quotient $\text{Cl}_{\text{loc}}(X) = \text{Cl}(X)/\text{Pic}(X)$. When this group is uniformly annihilated by D , every Weil divisor on X is \mathbb{Q} -Cartier of index bounded above by D . To revisit an example where the class group distinction is relevant for computing the smallest D that works, note that if $X = \text{Spec}(\mathbb{Z}[\sqrt{-5}])$, then $\text{Pic}(X) = \text{Cl}(X) \cong \mathbb{Z}/2\mathbb{Z}$, so $\text{Cl}_{\text{loc}}(X) = 0 \neq \text{Cl}(X)$. Indeed, **any** abelian group arises up to isomorphism as the divisor class group $\text{Cl}(\text{Spec}(R))$ for some Dedekind domain R (cf., Claborn [4] and Example 6.3.2 in [9], II.6), while $\text{Cl}_{\text{loc}}(\text{Spec}(R)) = 0$ for any Dedekind domain R and its order $D = 1$ is the optimal multiplier noted earlier. These observations suggest that in computing the optimal uniform annihilator D , we should emphasize the local class group over the global class group. Furthermore, there is an exact sequence of \mathbb{Z} -modules,

$$0 \rightarrow \text{Pic}(X) \rightarrow \text{Cl}(X) \xrightarrow{\phi} M := \prod_{\mathfrak{m}} \text{Cl}(R_{\mathfrak{m}}),$$

where \mathfrak{m} runs through all maximal ideals in R , and where the tuple $\phi(a_1[P_1] + \cdots + a_r[P_r])$ has \mathfrak{m} -th coordinate $a_1[P_1R_{\mathfrak{m}}] + \cdots + a_r[P_rR_{\mathfrak{m}}]$. For *nontrivial* effective Weil divisors, this rule translates in terms of ideals to $\phi([P_1^{(a_1)} \cap \cdots \cap P_r^{(a_r)}]) = ([P_1^{(a_1)}R_{\mathfrak{m}} \cap \cdots \cap P_r^{(a_r)}R_{\mathfrak{m}}])_{\mathfrak{m}} \in M$.

In particular, $\text{Cl}_{\text{loc}}(X)$ is (isomorphic to) a subgroup of M . When R is uniformly torsion with multiplier D , we infer that M (and hence $\text{Cl}_{\text{loc}}(X)$) has the property that $D \cdot g = 0$ for all $g \in M$, and hence is uniformly annihilated with multiplier D .¹ This uniformity condition on a group is stronger than being **torsion**, meaning that every group element has finite order: for instance, the infinitely-generated abelian group $G = \bigoplus_{n \geq 2} (\mathbb{Z}/n\mathbb{Z})$ is torsion but not uniformly annihilated (indeed, $\text{Ann}_{\mathbb{Z}}(G) = 0\mathbb{Z}$). However, a finitely-generated abelian group is uniformly annihilated (respectively, is torsion) if and only if it is finite.

To conclude our extended remark, we now record without proof a fact that will underpin our presentation in Section (3), where we can forego the global- versus local class group distinction. Here, $X = \text{Spec}(R)$ is a scheme (see Lemma 5.1 of Murthy [21] for the \mathbb{N} -graded case, where it is noted that $\text{Cl}(R) \cong \text{Cl}(R_{\mathfrak{m}})$, where \mathfrak{m} is the sole homogeneous maximal ideal; the local case follows since being a locally free module is equivalent to being a free module over a local ring):

(F): If the normal Noetherian domain (R, \mathfrak{m}) is additionally either local, or \mathbb{N} -graded local over a field, then $\text{Pic}(X)$ is trivial (i.e., every projective R -module of constant rank 1 is free), and hence the local class group $\text{Cl}_{\text{loc}}(X) \cong \text{Cl}(X)$ is the global class group, up to canonical isomorphism.

2.2. The Main Lemma. Symbolic powers are notoriously difficult to compute by hand, i.e., to give generators for. Working in height one, item (a) of the following proposition allows us to study them indirectly. Note that if I is a proper ideal in a Noetherian ring, we say that I has **pure height** h if all of its associated primes have height h , in particular, none are embedded.

Proposition 2.2. *Let R be a Noetherian normal domain of positive Krull dimension, and \mathfrak{q} any ideal of pure height one with associated primes P_1, \dots, P_r . Then:*

- (a) *There exist positive integers b_1, \dots, b_r , uniquely determined by \mathfrak{q} , such that the symbolic power $\mathfrak{q}^{(E)} = P_1^{(Eb_1)} \cap \dots \cap P_r^{(Eb_r)}$ for all $E > 0$.*
- (b) *If either (1) $D \cdot \text{Cl}(R) = 0$, or (2) the class $[\mathfrak{q}] \in \text{Cl}(R)$ has finite order D , then for all integers $a > 0$, $\mathfrak{q}^{(Da)} = (\mathfrak{q}^{(D)})^a$ is principal and $\mathfrak{q}^{(Da)} \subseteq \mathfrak{q}^a$.*

Proof. First, we prove (a). Recall that according to ([2], Proposition 4.9), $\mathfrak{q}^{(1)} = \mathfrak{q}$. For each i , the local ring $S_i = R_{P_i}$ is a discrete valuation ring, and we let $t_i \in S_i$ be a local uniformizing parameter. Then S_i is a PID, so an ideal $J \subseteq S_i$ is $P_i S_i = (t_i) S_i$ -primary if and only if $J = (P_i S_i)^n = P_i^n S_i = (t_i^n) S_i$ for some $n > 0$. In particular, $\mathfrak{q} S_i$ is $P_i S_i$ -primary, say $\mathfrak{q} S_i = (t_i^{b_i}) S_i$. Then the P_i -primary component of \mathfrak{q} is $P_i^{(b_i)}$. Thus $\mathfrak{q} = P_1^{(b_1)} \cap \dots \cap P_r^{(b_r)}$ and clearly the b_i are uniquely determined by \mathfrak{q} . Similarly, $\mathfrak{q}^E S_i = (t_i^{Eb_i}) S_i$ for $E > 0$, so $\mathfrak{q}^{(E)} = P_1^{(Eb_1)} \cap \dots \cap P_r^{(Eb_r)}$ for all $E > 0$. Thus, we may define divisor classes $[\mathfrak{q}] := b_1[P_1] + \dots + b_r[P_r] \in \text{Cl}(R)$ and more generally $[\mathfrak{q}^{(E)}] := E[\mathfrak{q}] = Eb_1[P_1] + \dots + Eb_r[P_r] \in \text{Cl}(R)$ for each $E > 0$; such a class vanishes if and only if $\mathfrak{q}^{(E)}$ is principal (see Hochster's lecture notes [12] for a proof).²

For (b), the argument is the same assuming (1) or (2). Continuing from (a), suppose that $\mathfrak{q}^{(E)} = P_1^{(Eb_1)} \cap \dots \cap P_r^{(Eb_r)}$ for all $E > 0$ and some integers $b_1, \dots, b_r > 0$ uniquely determined by \mathfrak{q} . Notice that each symbolic power $\mathfrak{q}^{(E)} \subseteq \mathfrak{q}^{(1)} = \mathfrak{q}$, since $I^{(n)} \subseteq I^{(m)}$ for each pair $n \geq m$ and any ideal $I \subseteq R$. Both (1) and (2) imply $\mathfrak{q}^{(D)} \subseteq \mathfrak{q}$ is principal. By taking a -th powers, the proposition follows in full once we explain how $\mathfrak{q}^{(Da)} = (\mathfrak{q}^{(D)})^a$ for all integers $a > 0$. Indeed, if

¹In group theory, any D serving such a role is called an **exponent** of the group, when written multiplicatively; since we understand \mathbb{Z} -modules additively, we call D a multiplier instead.

²Taking E to be the order of the class $[\mathfrak{q}]$, we can characterize the classes $[\mathfrak{q}]$ in the torsion subgroup of $\text{Cl}(R)$ in terms of those ideals \mathfrak{q} of pure height one having some principal symbolic power.

$J = (f) \subsetneq R$ is a nonzero, proper principal ideal in a Noetherian normal domain, it is of pure height one and has a unique minimal primary decomposition of the form $(f) = \bigcap_{i=1}^L Q_i^{(B_i)}$, where the Q_i are the minimal primes of J , all height one by the Krull principal ideal theorem, and B_i is the order of f in the discrete valuation ring R_{Q_i} (see Thm 11.5 of Matsumura [20], and [12]). Therefore, since discrete valuations are group homomorphisms, $J^a = (f^a) = \bigcap_{i=1}^L Q_i^{(aB_i)}$ for each integer $a > 0$. Now simply set $J = \mathfrak{q}^{(D)}$, $L = r$, $Q_i = P_i$, and $B_i = Db_i$. \square

In particular, when condition (a) holds, the multiplier D depends only on the ring R and not on the ideal $\mathfrak{q} \subseteq R$ of pure height one.

We now prove the main lemma of the paper, which expands upon Lemma (1.1):

Lemma 2.3. *Suppose a reduced Noetherian ring R is uniformly annihilated with multiplier D . Then for all $a > 0$ and all ideals $\mathfrak{q} \subseteq R$ of pure height one, the symbolic power $\mathfrak{q}^{(Da)} \subseteq \mathfrak{q}^a$. When $D := D_{\min}(R)$ is assumed to be minimal among uniform annihilators for R , the bound D is best possible for obtaining these inclusions across all R .*

Remark. $D_{\min}(R)$ can be understood as follows. Since R is uniformly annihilated, for a fixed maximal ideal $\mathfrak{m} \subseteq R$ the annihilator ideal $\text{Ann}_{\mathbb{Z}}(\text{Cl}(R_{\mathfrak{m}})) = E_{\mathfrak{m}}\mathbb{Z}$ where $E_{\mathfrak{m}}$ is simply the least common multiple (LCM) of the orders of all $g \in \text{Cl}(R_{\mathfrak{m}})$. In turn, since R is uniformly annihilated, the LCM of the $E_{\mathfrak{m}}$ is well-defined and $D_{\min}(R)$ is simply this LCM.

Proof of Lemma (2.3). First, we reduce to the local case. Indeed, recall that given two ideals I, J in R , the inclusion $I \subseteq J$ holds (that is, the R -module $\frac{J+I}{J} = 0$) if and only if $IR_{\mathfrak{m}} \subseteq JR_{\mathfrak{m}}$ (that is, the $R_{\mathfrak{m}}$ -module $\frac{(J+I)R_{\mathfrak{m}}}{JR_{\mathfrak{m}}} = \frac{JR_{\mathfrak{m}}+IR_{\mathfrak{m}}}{JR_{\mathfrak{m}}} = 0$) for all maximal ideals $\mathfrak{m} \subseteq R$. So we may assume R is a normal Noetherian local domain (in keeping with R being uniformly annihilated), whence $\text{Cl}_{\text{loc}}(R) = \text{Cl}(R)$ by Fact (F) of the preceding subsection. Then, for R , D , and all \mathfrak{q} as stated, Proposition (2.2)(b) gives us the inclusions.

Returning to the more general setting of the lemma, our remarks about Dedekind domains indicate that the LCM $D_{\min}(R)$ is best possible for obtaining these inclusions across all R . \square

To make a final remark in passing, suppose that the normal Noetherian domain (R, \mathfrak{m}) is local or graded local over a field, and that $G = \text{Cl}(R)$ is finite. Then $D = D_{\min}(R)$ is the order of G if and only if G is cyclic. This holds simply because D is the maximal order of a cyclic summand in the invariant decomposition isomorphism type of G as a \mathbb{Z} -module. In particular, there is a class $h \in G$ of order D , and any such h generates a cyclic subgroup of G of maximal cardinality.

3. EFFECTIVE UNIFORM BOUNDS

We begin with a selective review of toric geometry, for the benefit of non-experts. Readers familiar with the material can skip to subsection (3.1).

Affine Toric Algebraic Geometry. For a more in-depth tour of the results we need from the theory of normal toric varieties up through divisor theory (including exercises and omitted definitions and proofs, when given), we refer the reader to: sections 1.2, 1.3, 3.1, 3.2, 4.1, 4.2 of Cox, Little, and Schenck ([5]); or alternatively, Chapter 1 and sections 3.1, 3.3, 3.4 of Fulton

([8]).³ For simplicity, we conduct our review with the standard lattice \mathbb{Z}^n in \mathbb{R}^n , and a fixed algebraically closed field k .⁴

Any normal affine toric n -fold can be obtained from a strongly convex, rational polyhedral cone σ in \mathbb{R}^n ; we generally abbreviate this, saying only that σ is (SC-R). First, a **polyhedral cone** in \mathbb{R}^n is a closed, convex set

$$\sigma = \text{Cone}(G) = \left\{ \sum_{v \in G} a_v \cdot v : \text{each } a_v \in \mathbb{R}_{\geq 0} \right\} \subseteq \mathbb{R}^n,$$

generated by a finite set G (possibly empty) of nonzero vectors. The strong convexity (SC) condition is simply that in addition to being convex, σ contains no line through the origin. The rationality (R) condition is simply that $G \subseteq \mathbb{Z}^n$. Note that the cone has **dimension** $\dim \sigma := \dim(\mathbb{R}\text{-linear span of } G) \leq n$; if $\dim \sigma = n$, then σ has maximal dimension. In particular, the **zero(-dimensional) cone** $\{0\} = \text{Cone}(\emptyset)$ consists only of the origin. Since $\sigma = \text{Cone}(G) \subseteq \mathbb{R}^n$ is rational, its **dual** polyhedral cone

$$\sigma^\vee := \{w \in \mathbb{R}^n : \langle w, v \rangle \geq 0 \text{ for all } v \in G\}$$

is also rational with respect to \mathbb{Z}^n , where $\langle \cdot, \cdot \rangle$ is dot product. This rational cone σ^\vee in turn yields

- (1) A finitely-generated semigroup under addition

$$(S_\sigma, +) := \sigma^\vee \cap \mathbb{Z}^n = \mathbb{Z}_{\geq 0}[m_1, \dots, m_r] = \left\{ \sum_{i=1}^r a_i m_i : \text{each } a_i \in \mathbb{Z}_{\geq 0} \right\}$$

for some finite list of generators $m_1, \dots, m_r \in \mathbb{Z}^n$.

- (2) A normal domain of finite type over k : namely, the semigroup ring $R = k[S_\sigma]$ generated as an algebra by the characters χ^m with $m \in S_\sigma$. If $S_\sigma = \mathbb{Z}_{\geq 0}[m_1, \dots, m_r]$, then $R = k[\chi^{m_1}, \dots, \chi^{m_r}]$ as an algebra. In computations, we regard $\chi^a = t_1^{a_1} \cdots t_n^{a_n}$ ($a = (a_1, \dots, a_n)$) as a Laurent monomial in n variables, whence R is a subring of the domain $k[S_{\{0\}}] = k[\mathbb{Z}^n] \cong k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]$ of Laurent polynomials in n variables over k .

Any domain so obtained is a **toric ring** (or normal affine semigroup algebra) over k , and can be graded by \mathbb{Z}^n ($\deg(\chi^{m_i}) = m_i$) so that $R_0 = k$ ($0 \in \mathbb{Z}^n$) and there is a unique homogeneous maximal ideal. In ([11]), Hochster shows that over any field, normal semigroup rings are Cohen-Macaulay. Therefore, toric rings are Cohen-Macaulay.

- (3) A normal toric n -fold, assuming σ is SC-R: Define $U_\sigma = \text{Specm}(R)$, a normal affine variety. (Specm consists of the closed points of Spec .) Strong convexity ensures that the n -torus $(k^\times)^n = \text{Specm}(k[\mathbb{Z}^n])$ embeds as a dense open set in U_σ , whence

$$k(U_\sigma) = \text{frac}(R) = \text{frac}(k[t_1^{\pm 1}, \dots, t_n^{\pm 1}]) = k(t_1, \dots, t_n) \Rightarrow \dim(U_\sigma) = \dim(R) = n.$$

An (SC-R) cone $\sigma \subseteq \mathbb{R}^n$ is **simplicial** if $\sigma = \text{Cone}(G)$ for some $G \subseteq \mathbb{Z}^n$ forming part of a \mathbb{R} -basis for \mathbb{R}^n , and we call the corresponding toric ring and toric variety **simplicial**. In this case, σ has $\binom{\dim(\sigma)}{\ell}$ faces of dimension ℓ .⁵ Moreover, we may assume $G = \{v_1, \dots, v_{\dim(\sigma)}\}$ consists of

³While both references work over \mathbb{C} , the facts we review in this paper hold over any algebraically closed field. Indeed, the method of constructing semigroup rings and toric varieties from cones works over any field. Omitted proofs of the reviewed bijective correspondences and short exact sequence depend, at least implicitly, on Lemma 1.3.1 of ([5]), whose proof in turn requires the Nullstellensatz. See footnote 6 below.

⁴In alignment with the literature, and adjusting the presentation accordingly, one could instead fix an arbitrary lattice N considered in the Euclidean space $N_{\mathbb{R}} = N \otimes_{\mathbb{Z}} \mathbb{R}$, rendering it to be the standard lattice in that Euclidean space. The latter has dual vector space $N_{\mathbb{R}}^*$, and character lattice $M = \text{Hom}_{\mathbb{Z}}(N, \mathbb{Z}) \subseteq N_{\mathbb{R}}^*$ dual to N , and M is the standard lattice in $N_{\mathbb{R}}^*$.

⁵Roughly, a face is a polyhedral subcone of σ obtained by intersecting σ with any appropriately-chosen supporting hyperplane in \mathbb{R}^n . If σ is (SC-R), then so are all its (finitely-many) faces.

primitive vectors, i.e., that the coordinates of each v_i have no common prime integer factor. When σ is n -dimensional, we may index the v_i so that the matrix $A_G \in \text{Mat}_{n \times n}(\mathbb{Z})$ whose i -th row is v_i has determinant $d = \det(A_G) \in \mathbb{Z}_{>0}$. Note that d is the volume of the n -parallelotope spanned by v_1, \dots, v_n .

We now arrive at a point where k being algebraically closed is useful. Relative to a fixed SC-R cone $\sigma \subseteq \mathbb{R}^n$, and without being explicit, the faces τ of σ are in bijection with the following sets:⁶

- $\{\text{torus-orbits of } U_\sigma\}$
- $\{\text{torus-invariant closed subvarieties of } U_\sigma\}$

via $\tau \mapsto O(\tau) \mapsto V(\tau) = \overline{O(\tau)}$ (Zariski closure in U_σ), where according to Lemma 3.2.5 of ([5]), if $\tau^\perp \cap \mathbb{Z}^n := \{m \in \mathbb{Z}^n : \langle m, v \rangle = 0 \text{ for all } v \in \tau\}$, then

$$O(\tau) := \{\text{semigroup homomorphisms } \gamma : (S_\tau, +) \rightarrow (k, \times) : \gamma(m) \neq 0 \iff m \in \tau^\perp \cap \mathbb{Z}^n\},$$

$$\cong \text{Hom}_{\mathbb{Z}}(\tau^\perp \cap \mathbb{Z}^n, k^\times).$$

is a $\text{codim}(\tau)$ -torus, where $\text{codim}(\tau) = \dim \sigma - \dim \tau$. Thus we note that $\dim(V(\tau)) = \text{codim}(\tau)$.

Going forward, let $\Sigma(1)$ denote the collection of **rays** (one-dimensional faces).

3.0.1. Divisor Theory. In particular, each ray $\rho \in \Sigma(1)$ has a generator $u_\rho \in \rho \cap \mathbb{Z}^n$ that is primitive, and yields a torus-invariant prime divisor $D_\rho = V(\rho)$ on $X = U_\sigma$. In turn, each D_ρ ($\rho \in \Sigma(1)$) gives a DVR \mathcal{O}_{X, D_ρ} with valuation $v_\rho = v_{D_\rho} : k(X)^\times \rightarrow \mathbb{Z}$, where $k(X)^\times$ consists of the nonzero rational functions on X . Any torus-invariant Weil divisor on X has the form $\sum_{\rho \in \Sigma(1)} a_\rho D_\rho$ with each $a_\rho \in \mathbb{Z}$ (cf. Exercise 4.1.1 of [5]), and conversely. Thus

$$\text{Div}_T(X) = \bigoplus_{\rho \in \Sigma(1)} \mathbb{Z} D_\rho$$

is the free abelian group of torus-invariant Weil divisors on X , and we close our review by situating without proof a few facts from sections 4.1 and 4.2 of ([5]) about the divisor class group $\text{Cl}(X)$, specialized to the affine case.

Theorem 3.1. *Take a normal affine toric variety $X = U_\sigma$. Then*

- (1) $\text{Pic}(X) = 0$ (cf., Proposition 4.2.2 of [5]), so $\text{Cl}_{\text{loc}}(X) \cong \text{Cl}(X)$.
- (2) $\text{Cl}(X)$ is finite abelian if and only if σ is simplicial (cf., Proposition 4.2.7 of [5]). If so, then all Weil divisors on X are \mathbb{Q} -Cartier of index bounded by the order of $\text{Cl}(X)$.
- (3) If $G = \{u_\rho : \rho \in \Sigma(1)\}$ spans \mathbb{R}^n , then the following sequence of abelian groups is exact

$$0 \rightarrow \mathbb{Z}^n \xrightarrow{\phi} \text{Div}_T(X) \rightarrow \text{Cl}(X) \rightarrow 0,$$

where $\phi(m) = \text{div}(\chi^m) = \sum_{\rho \in \Sigma(1)} \langle m, u_\rho \rangle D_\rho$, and $\langle \cdot, \cdot \rangle$ is dot product.

With the review complete, we proceed to deduce effective uniform bounds in the toric setting. We expand on Theorem (3.1)(2) in case σ has **maximal** dimension, giving a quick proof of Theorem (1.3) using the short exact sequence (3.1)(3).

⁶This is a special case of the Orbit-Cone Correspondence (Theorem 3.2.6 of [5]). The proof of the latter uses the bijective correspondence between semigroup homomorphisms $S_\sigma \rightarrow \mathbb{C}^\times$ and points on the toric variety U_σ . In turn, the proof of this correspondence (Lemma 1.3.1) uses the Nullstellensatz.

3.1. The Toric Case. The short exact sequence (3.1)(3) makes it easy to compute the divisor class group $\text{Cl}(U_\sigma)$, via a Smith normal form approach that is somewhat indirect. If $\Sigma(1) = \{\rho_1, \dots, \rho_r\}$ has size r ,⁷ the map ϕ can be treated, up to isomorphism, as a map $\mathbb{Z}^n \rightarrow \mathbb{Z}^r$ given by the matrix $A = (u_{\rho_1}, \dots, u_{\rho_r})^T$ where T denotes transpose: that is, the i -th row of A is given by the coordinates of u_{ρ_i} . $\text{Cl}(U_\sigma)$ is the cokernel of ϕ , and hence can be computed up to isomorphism by first finding the Smith normal form of A . Note that since the alternating sum of the ranks in the short exact sequence vanishes, $\text{Cl}(U_\sigma)$ has rank $r - n$. We now re-express Theorem (1.3):

Theorem 3.2. *Take $\sigma = \text{Cone}(G) \subseteq \mathbb{R}^n$ simplicial of maximal dimension, and $R = k[U_\sigma]$ the corresponding semigroup ring. Let $A_G \in \text{Mat}_{n \times n}(\mathbb{Z})$ be the accompanying matrix of primitive ray generators, and $d = \det(A_G) > 0$ the volume of the n -parallelotope spanned by the u_ρ ($\rho \in \Sigma(1)$). Then $\text{Cl}(U_\sigma)$ is finite abelian of order d .*

Proof. We maintain the notational conventions of the statement of Theorem (3.1), along with those in the paragraph on simplicial toric varieties in our review above. By hypothesis, $\Sigma(1)$ consists of n rays, and $G = \{u_\rho : \rho \in \Sigma(1)\}$ forms an \mathbb{R} -vector space basis of \mathbb{R}^n since σ is simplicial. So by Theorem (3.1)(3), $\text{Cl}(U_\sigma)$ has rank zero, and hence is finite abelian. Moreover, as a special case of following the Smith normal form approach in the paragraph above, we see that the matrix A_G defines the action of ϕ . Thus we conclude that d is the order of $\text{Cl}(U_\sigma)$. This also confirms Theorem (1.3), since the order is part of the hypotheses of Lemma (1.1). \square

3.2. Rational double points. To obtain additional explicit, effective multipliers, we turn to the case of complete, normal Noetherian local domains S in equal characteristic zero with du Val (ADE) isolated singularity and algebraically closed residue field; for simplicity, we work with \mathbb{C} . In ([19], §24), Lipman computes the class group isomorphism type (as a \mathbb{Z} -module) of each du Val singularity. The du Val (ADE) singularities, also known as *rational double points*, are the most basic isolated surface singularities. For a brief introduction to these singularities and their basic properties, we refer the reader to sections III.3 and III.7 of ([3]). Their resolutions can be understood and classified by the simply-laced Dynkin diagrams of types A, D, and E. We can express S as above as the quotient of the power series $\mathbb{C}[[x, y, z]]$ by a single local equation. We now situate a succinct data table for the du Val singularities of each type, where the last column's entries depend on our closing remark about cyclic subgroups of finite class groups in Section (2):

Singularity type	Local Equation	Class group (isomorphism type)	$D_{\min}(S)$
A_n ($n \geq 1$)	$xz - y^{n+1}$	$\mathbb{Z}/(n+1)\mathbb{Z}$	$n+1$
D_n ($n \geq 4$)	$x^2 + yz^2 - z^{n-1}$	$\begin{cases} (\mathbb{Z}/2\mathbb{Z})^2 & n \text{ even} \\ \mathbb{Z}/4\mathbb{Z} & n \text{ odd.} \end{cases}$	$\begin{cases} 2 & n \text{ even} \\ 4 & n \text{ odd.} \end{cases}$
E_n ($n = 6, 7, 8$)	$\begin{cases} x^4 + y^3 + z^2 & \text{if } n = 6 \\ x^3y + y^3 + z^2 & \text{if } n = 7 \\ x^5 + y^3 + z^2 & \text{if } n = 8 \end{cases}$	$\begin{cases} \mathbb{Z}/3\mathbb{Z} & \text{if } n = 6 \\ \mathbb{Z}/2\mathbb{Z} & \text{if } n = 7 \\ 0 & \text{if } n = 8 \end{cases}$	$\begin{cases} 3 & \text{if } n = 6 \\ 2 & \text{if } n = 7 \\ 1 & \text{if } n = 8 \end{cases}$

Note that an analogous data table can be drafted for affine du Val singularity hypersurfaces in \mathbb{C}^3 (affine, \mathbb{N} -graded local case). These hypersurfaces are obtained as the quotients \mathbb{C}^2/G of affine 2-space by the action of a finite subgroup $G \subseteq SU_2(\mathbb{C})$.

Taking the above data table in tandem with Theorem (1.3), we now have access to an infinite supply of concrete, effective uniform bounds that hold in **non-regular** Noetherian domains

⁷Note that $r \geq n$ since G spans \mathbb{R}^n by hypothesis in Theorem (3.1)(3).

whose singularities are sufficiently nice (toric, du Val). In particular, the Huneke-Katz-Validashti result ([16]) on uniform bounds now has some added company in the setting of rational surface singularities, courtesy of Lipman's work.

Remark. When a Noetherian ring R satisfies the uniform symbolic topology property (USTP) on prime ideals, experts might initially expect that the optimal multiplier $D = D_{\min}(R)$ should depend only on simple numerical invariants of R , such as Krull dimension, or the multiplicity of R at an isolated singularity. However, A_n -singularities and E_8 -singularities have multiplicity two, being rational double points. At one extreme, $D_{\min}(A_n) = n + 1$ is optimal, grows arbitrarily large with n , and does **not** depend on any such numerical invariants of A_n . At the other extreme, $D_{\min}(E_8) = 1$ is lower than both the Krull dimension and the multiplicity. Therefore, a (sharp) uniform bound depending only on such numerical invariants need not exist.

4. LINGERING QUESTIONS AND FUTURE DIRECTIONS

To summarize, we have deduced a group-theoretic criterion (Lemma (2.3)) for uniform linear equivalence of symbolic- and adic topologies of ideals of pure height one in a Noetherian normal domain of positive Krull dimension. We also demonstrated the criterion's utility relative to familiar classes of local- or graded local Cohen-Macaulay domains with rational singularities (e.g., all toric rings and two-dimensional Noetherian normal domains are Cohen-Macaulay [10]). We close by briefly mentioning two natural lines for further investigation.

- (1) Under the hypotheses on R and $\text{Cl}(R)$ in Lemma (2.3), can the lemma be strengthened: in particular, can we cast off the stipulation that the ideals have pure height one in deducing containments, instead allowing any ideal of height one? Our current arguments rely on this stipulation.
- (2) If we restrict attention to symbolic powers of prime ideals, can we identify a candidate mechanism (i.e., group-theoretic) to verify the uniform symbolic topology property for prime ideals of height two or more, in nice classes of (non-regular) Noetherian normal domains of dimension at least three?

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